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Sharpening Redheffer-type inequalities for circular functions

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ABSTRACT

In this note, some new sharpened Redheffer-type inequalities involving circular functions are established.

1. Introduction

Redheffer [1] posed the problem of proving the inequality

$$\frac{\sin x}{x} \geq \frac{\pi^2 - x^2}{\pi^2 + x^2}, \quad x \in (0, \pi]. \tag{1}$$

Williams [2] proved the inequality (1). Chen, Zhao, and Qi [3] obtained three Redheffer-type inequalities for $\cos x$, $\cosh x$, and $\sinh x/x$ using the infinite product representations of $\cos x$, $\cosh x$, and $\sinh x$; the first one is

$$\cos x \geq \frac{\pi^2 - 4x^2}{\pi^2 + 4x^2}, \quad x \in \left[0, \frac{\pi}{2}\right]. \tag{2}$$

Zhu and Sun [4] extended and sharpened inequalities (1) and (2) above, and showed a new Redheffer-type inequality for $\tan x$ as follows:

Theorem 1. *Let $0 < x < \pi$. Then*

$$\left(\frac{\pi^2 - x^2}{\pi^2 + x^2}\right)^{\beta} \leq \frac{\sin x}{x} \leq \left(\frac{\pi^2 - x^2}{\pi^2 + x^2}\right)^{\alpha} \tag{3}$$

holds if and only if $\alpha \leq \pi^2/12$ and $\beta \geq 1$.

Theorem 2. *Let $0 \leq x \leq \pi/2$. Then*

$$\left(\frac{\pi^2 - 4x^2}{\pi^2 + 4x^2}\right)^{\beta} \leq \cos x \leq \left(\frac{\pi^2 - 4x^2}{\pi^2 + 4x^2}\right)^{\alpha} \tag{4}$$

holds if and only if $\alpha \leq \pi^2/16$ and $\beta \geq 1$.

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Theorem 3. *Let* $0 < x < \pi/2$. *Then*

$$\left(\frac{\pi^2 + 4x^2}{\pi^2 - 4x^2}\right)^{\alpha} \le \frac{\tan x}{x} \le \left(\frac{\pi^2 + 4x^2}{\pi^2 - 4x^2}\right)^{\beta} \tag{5}$$

holds if and only if $\alpha \le \pi^2/24$ *and* $\beta \ge 1$.

Recently, Li and Li [5] give a new Redheffer-type inequality about an upper bound for $\sin x/x$:

$$\frac{\sin \pi x}{\pi x} \le \frac{1 - x^2}{\sqrt{1 + 3x^4}}, \quad x \in (0, 1], \tag{6}$$

that is

$$\frac{\sin x}{x} \le \frac{\pi^2 - x^2}{\sqrt{\pi^4 + 3x^4}}, \quad x \in (0, \pi]. \tag{7}$$

Combining inequalities (1) and (7) gives

$$\frac{\pi^2 - x^2}{\pi^2 + x^2} \le \frac{\sin x}{x} \le \frac{\pi^2 - x^2}{\sqrt{\pi^4 + 3x^4}}, \quad x \in (0, \pi]. \tag{8}$$

In the form of (8), some new Redheffer-type inequalities are shown by Zhu [6], described as Theorems 4 and 5 for $\cos x$ and $\tan x$, as follows.

Theorem 4. *Let* $0 < x \le \pi/2$. *Then*

$$\frac{\pi^2 - 4x^2}{\pi^2 + 4x^2} \le \cos x \le \left(\frac{\pi^2 - 4x^2}{\sqrt{\pi^4 + 48x^4}}\right)^{3/4} \tag{9}$$

holds.

Theorem 5. *Let* $0 < x < \pi/2$. *Then*

$$\left(\frac{\sqrt{\pi^4 + 48x^4}}{\pi^2 - 4x^2}\right)^{1/2} \le \frac{\tan x}{x} \le \frac{\pi^2 + 4x^2}{\pi^2 - 4x^2} \tag{10}$$

holds.

In this note, we sharpen inequality (7) in exponential type, and sharpen inequality (9) and (10) in the same way via two new exponential-type inequalities described as Lemmas 3 and 4 for circular functions.

Theorem 6. *Let* $0 < x \le \pi$. *Then*

$$\left(\frac{\pi^2 - x^2}{\sqrt{\pi^4 + 3x^4}}\right)^{\alpha} \le \frac{\sin x}{x} \le \left(\frac{\pi^2 - x^2}{\sqrt{\pi^4 + 3x^4}}\right)^{\beta} \tag{11}$$

holds if and only if $\alpha \ge \pi^2/6$ *and* $\beta \le 1$.

Theorem 7. *Let* $0 < x \le \pi/2$. *Then*

$$\left(\frac{\pi^2 - 4x^2}{\sqrt{\pi^4 + 48x^4}}\right)^{\pi^2/6} \le \cos x \le \left(\frac{\pi^2 - 4x^2}{\sqrt{\pi^4 + 48x^4}}\right)^{3/4} \tag{12}$$

holds.

Theorem 8. *Let* $0 < x < \pi/2$. *Then*

$$\left(\frac{\sqrt{\pi^4 + 48x^4}}{\pi^2 - 4x^2}\right)^{1/2} \le \frac{\tan x}{x} \le \left(\frac{\sqrt{\pi^4 + 48x^4}}{\pi^2 - 4x^2}\right)^{\pi^2/6} \tag{13}$$

holds.

2. Nine lemmas

Lemma 1 *([7–10]). Let $f, g : [a, b] \to \mathbb{R}$ be two continuous functions which are differentiable on (a, b). Further, let $g' \neq 0$ on (a, b). If f'/g' is increasing (or decreasing) on (a, b), then the functions $(f(x)-f(b))/(g(x)-g(b))$ and $(f(x)-f(a))/(g(x)-g(a))$ are also increasing (or decreasing) on (a, b).*

Lemma 2 *([11–13]). Let a_n and b_n $(n = 0, 1, 2, \ldots)$ be real numbers, and let the power series $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ be convergent for $|x| < R$. If $b_n > 0$ for $n = 0, 1, 2, \ldots$, and if a_n/b_n is strictly increasing (or decreasing) for $n = 0, 1, 2, \ldots$, then the function $A(x)/B(x)$ is strictly increasing (or decreasing) on $(0, R)$.*

Lemma 3. *Let $0 < x \leq \pi/2$. Then*

$$\left(\frac{\sin 2x}{2x}\right)^{\alpha} \leq \cos x \leq \left(\frac{\sin 2x}{2x}\right)^{\beta} \tag{14}$$

holds if and only if $\alpha \geq 1$ and $\beta \leq 3/4$.

Lemma 4. *Let $0 < x < \pi/2$. Then*

$$\left(\frac{2x}{\sin 2x}\right)^{\alpha} \leq \frac{\tan x}{x} \leq \left(\frac{2x}{\sin 2x}\right)^{\beta} \tag{15}$$

holds if and only if $\alpha \leq 1/2$ and $\beta \geq 1$.

Lemma 5 *([14, Theorem 3.4]). Let B_{2n} be the even-indexed Bernoulli numbers, and $\zeta(\cdot)$ the Riemann zeta function. Then*

$$\zeta(2n) = \frac{(2\pi)^{2n}}{2(2n)!}|B_{2n}|, \quad n = 1, 2, \ldots. \tag{16}$$

(For further information on the even-indexed Bernoulli numbers B_{2n}, refer to pp. 231–232 in [15].)

Lemma 6 *([16]). Let us have integers $n \geq 1$ and B_{2n} the even-indexed Bernoulli numbers. Then*

$$|B_{2n}| \leq \frac{2(2n)!}{(2\pi)^{2n}} \frac{1}{1 - 2^{\lambda-2n}} \tag{17}$$

holds, where $\lambda = 2 + \frac{\log(1-6/\pi^2)}{\log 2}$.

Lemma 7. *Let us have integers $n \geq 2$ and $\zeta(\cdot)$ the Riemann zeta function. Then*

$$\zeta(2n+2) + 3\zeta(2n-2) \leq \frac{2}{3}\pi^2 \tag{18}$$

holds.

Proof. By the relational expression (16), the inequality (18) is equivalent to the following one:

$$\frac{(2\pi)^{2n+2}}{2(2n+2)!}|B_{2n+2}| + \frac{3(2\pi)^{2n-2}}{2(2n-2)!}|B_{2n-2}| \leq \frac{2}{3}\pi^2. \tag{19}$$

Using inequality (17), (19) can be completed on proving the following result:

$$\frac{2^{2n+2}}{2^{2n+2} - 2^{\lambda}} + \frac{3 \cdot 2^{2n-2}}{2^{2n-2} - 2^{\lambda}} \leq \frac{2}{3}\pi^2, \quad n \geq 2, \tag{20}$$

where $\lambda = 2 + \frac{\log(1-6/\pi^2)}{\log 2}$. In fact, we can easily prove (20) using a basic differential method. □

Lemma 8. *Let $0 \leq x < \pi/2$. Then*

$$\tan x = \sum_{n=1}^{\infty} \frac{2(2^{2n}-1)}{\pi^{2n}} \zeta(2n) x^{2n-1}. \tag{21}$$

Proof. The following power series expansion can be found in [17, 1.3.1.4 (3)]:

$$\tan x=\sum_{n=1}^{\infty}\frac{2^{2n}(2^{2n}-1)}{(2n)!}(-1)^{n-1}B_{2n}x^{2n-1}=\sum_{n=1}^{\infty}\frac{2^{2n}(2^{2n}-1)}{(2n)!}|B_{2n}|x^{2n-1},\quad |x|<\frac{\pi}{2}.\tag{22}$$

Using the relational expression (16), we obtain (21). □

Lemma 9. *Let* $|x|<\pi$. *Then*

$$x\cot x=1-\sum_{n=1}^{\infty}\frac{2\zeta(2n)}{\pi^{2n}}x^{2n}.\tag{23}$$

Proof. The following power series expansion can be found in [17, 1.3.1.4 (2)]:

$$x\cot x=1-\sum_{n=1}^{\infty}\frac{2^{2n}}{(2n)!}|B_{2n}|x^{2n},\quad |x|<\pi.\tag{24}$$

Using the relational expression (16), we obtain (23). □

3. A concise proof of Lemma 3

In view of the fact that (14) holds for $x=\pi/2$, we set $0<x<\pi/2$ below.
Let $F(x)=\frac{\log\cos x}{\log\frac{\sin 2x}{2x}}=\frac{f_1(x)}{g_1(x)}$, where $f_1(x)=\log\cos x$, and $g_1(x)=\log\frac{\sin 2x}{2x}$. Then

$$\begin{aligned}\frac{f_1'(x)}{g_1'(x)}&=\frac{1}{2}\frac{2x(1-\cos 2x)}{\sin 2x-2x\cos 2x}=:\frac{1}{2}\frac{t(1-\cos t)}{\sin t-t\cos t}\\&=\frac{1}{2}\frac{f_2(t)}{g_2(t)},\end{aligned}$$

where $2x=t, f_2(t)=t(1-\cos t), g_2(t)=\sin t-t\cos t$, and $t\in(0,\pi)$, since

$$\frac{f_2'(t)}{g_2'(t)}=\frac{1-\cos t}{t\sin t}+1=:g(t)+1,$$

where $g(t)=\frac{1-\cos t}{t\sin t}=\frac{1}{t^2}(\frac{t}{\sin t}-t\cot t)$.
The power series expansion of the function $t/\sin t$ can be found in Li [18]:

$$\frac{t}{\sin t}=1+\sum_{n=1}^{\infty}\frac{2^{2n}-2}{(2n)!}|B_{2n}|t^{2n},\quad |t|<\pi.\tag{25}$$

By (25) and (24), we have

$$\begin{aligned}g(t)&=\sum_{n=1}^{\infty}\frac{2\cdot 2^{2n}-2}{(2n)!}|B_{2n}|t^{2n-2},\\g'(t)&=\sum_{n=2}^{\infty}\frac{2\cdot 2^{2n}-2}{(2n)!}|B_{2n}|(2n-2)t^{2n-3}>0\end{aligned}$$

for $t\in(0,\pi)$. So $g(t)$ is increasing on $(0,\pi)$, and $F(x)=\frac{f_1(x)}{g_1(x)}=\frac{f_1(x)-f_1(0^+)}{g_1(x)-g_1(0^+)}$ is increasing on $(0,\pi/2)$ by Lemma 1, repeatedly. Furthermore, $\lim_{x\to 0^+}F(x)=3/4$ and $\lim_{x\to(\pi/2)^-}F(x)=1$; the proof of Lemma 3 is complete.

4. A concise proof of Lemma 4

Let $G(x)=\frac{\log\frac{\tan x}{x}}{\log\frac{2x}{\sin 2x}}=\frac{f_1(x)}{g_1(x)}$, where $f_1(x)=\log\frac{\tan x}{x}$, and $g_1(x)=\log\frac{2x}{\sin 2x}$. Then

$$\frac{f_1'(x)}{g_1'(x)}=\frac{x\sec^2 x-\tan x}{x\sec^2 x+\tan x-2x}=:\frac{A(x)}{B(x)},$$

where $A(x)=x\sec^2 x-\tan x$ and $B(x)=x\sec^2 x+\tan x-2x$.

By (22), we have

$$\sec^2 x = (\tan x)' = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)}{(2n)!}(2n-1)|B_{2n}|x^{2n-2}, \quad |x| < \frac{\pi}{2}, \tag{26}$$

and

$$\begin{aligned} A(x) &= x\sec^2 x - \tan x = \sum_{n=2}^{\infty} \frac{2^{2n}(2^{2n}-1)}{(2n)!}(2n-2)|B_{2n}|x^{2n-1} \\ &=: \sum_{n=2}^{\infty} a_n x^{2n-1}, \quad |x| < \frac{\pi}{2}, \\ B(x) &= x\sec^2 x + \tan x - 2x = \sum_{n=2}^{\infty} \frac{2^{2n}(2^{2n}-1)}{(2n)!}2n|B_{2n}|x^{2n-1} \\ &=: \sum_{n=2}^{\infty} b_n x^{2n-1}, \quad |x| < \frac{\pi}{2} \end{aligned}$$

by (22) and (26), where $a_n = \frac{2^{2n}(2^{2n}-1)}{(2n)!}(2n-2)|B_{2n}|$ and $b_n = \frac{2^{2n}(2^{2n}-1)}{(2n)!}2n|B_{2n}| > 0$.

On setting $c_n = a_n/b_n$, we have $c_n = (2n-2)/(2n) = 1 - 1/n$ is increasing for $n = 2, 3, \ldots, A(x)/B(x)$ is increasing on $(0, \pi/2)$ and $\frac{f_1'(x)}{g_1'(x)}$ is increasing on $(0, \pi/2)$ by Lemma 2. Thus $G(x) = \frac{f_1(x)}{g_1(x)} = \frac{f_1(x)-f_1(0^+)}{g_1(x)-g_1(0^+)}$ is increasing on $(0, \pi/2)$ by Lemma 1. At the same time, $\lim_{x\to 0^+} G(x) = 1/2$ and $\lim_{x\to(\pi/2)^-} G(x) = 1$. So the proof of Lemma 4 is complete.

5. Proof of Theorem 6

Let $f(x) = \log\frac{\sin x}{x} - \frac{\pi^2}{6}\log\frac{\pi^2-x^2}{\sqrt{\pi^4+3x^4}}$. Then $f(0^+) = 0$, and

$$\begin{aligned} f'(x) &= \frac{x\cos x - \sin x}{x\sin x} - \frac{\pi^2}{6}\left[\frac{-2x}{\pi^2-x^2} - \frac{6x^3}{\pi^4+3x^4}\right] \\ &= \frac{1}{\pi^4+3x^4}\left[\pi^2 x^3 + \frac{\pi^2}{3}\frac{(\pi^4+3x^4)x}{\pi^2-x^2} - (\pi^4+3x^4)\frac{1-x\cot x}{x}\right]. \end{aligned}$$

By Lemma 9, we have

$$\begin{aligned} f'(x) &= \frac{1}{\pi^4+3x^4}\left[\pi^2x^3 + \frac{1}{3}(\pi^4+3x^4)\sum_{n=0}^{\infty}\left(\frac{x}{\pi}\right)^{2n}x^{2n+1} - 2(\pi^4+3x^4)\sum_{n=1}^{\infty}\frac{\zeta(2n)}{\pi^{2n}}x^{2n-1}\right] \\ &= \frac{1}{\pi^4+3x^4}\left(\frac{60\pi^2-\pi^4}{45}x^3 + 2\sum_{n=2}^{\infty}\frac{2\pi^2 - 3[\zeta(2n+2)+3\zeta(2n-2)]}{3\pi^{2n-2}}x^{2n+1}\right). \end{aligned}$$

By Lemma 7, we have $f'(x) > 0$ and $f(x)$ is increasing on $(0, \pi)$. Then $f(x) > f(0^+) = 0$ for $x \in (0, \pi)$, and the following inequality:

$$\left(\frac{\pi^2-x^2}{\sqrt{\pi^4+3x^4}}\right)^{\pi^2/6} \le \frac{\sin x}{x} \tag{27}$$

holds for $x \in (0, \pi)$.

By inequality (7) and inequality (27), we have a double inequality as follows:

$$\left(\frac{\pi^2-x^2}{\sqrt{\pi^4+3x^4}}\right)^{\pi^2/6} \le \frac{\sin x}{x} \le \frac{\pi^2-x^2}{\sqrt{\pi^4+3x^4}}, \quad x \in (0, \pi]. \tag{28}$$

Let $H(x) = \frac{\log\frac{\sin x}{x}}{\log\frac{\pi^2-x^2}{\sqrt{\pi^4+3x^4}}}$. Then $H(0^+) = \frac{\pi^2}{6}$, and $H(\pi^-) = 1$. So 1 and $\frac{\pi^2}{6}$ are the best constants in (28); the proof of Theorem 6 is complete.

6. A simple proof of Theorem 7

First, we give a simple proof of the left inequality of (12). By the left of the double inequality (14) when $\alpha = 1$ and inequality (27), we have

$$\cos x \geq \frac{\sin 2x}{2x} \geq \left(\frac{\pi^2 - 4x^2}{\sqrt{\pi^4 + 48x^4}}\right)^{\pi^2/6}.$$

Then, we obtain the light inequality of (12) by the right of the double inequality (14) when $\beta = 3/4$ and Redheffer-type inequality (7).

Remark. In the same way, we can easily obtain inequality (2).

7. A simple proof of Theorem 8

Using the left of the double inequality (15) when $\alpha = 1/2$ and Redheffer-type inequality (7) we can obtain the left of double inequality (13). At the same time, we give the right inequality of (13) by the right of the double inequality (15) when $\beta = 1$ and inequality (27).

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